TO KNOW AHEAD OF TIME

NUMERICAL SIMULATION IN POROUS MEDIA SYSTEMS

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Motivation

Subsidence of Ekofisk Oil Field

was a side effect of drilling.

Motivation

"They subsided so much they had to go in and raise the platforms, costing them several billion dollars. If they'd known ahead of time, they could have built their platforms taller"

Rick Dean, in "Modeling complex, multiphase porous media systems", Siam News, April 2002.

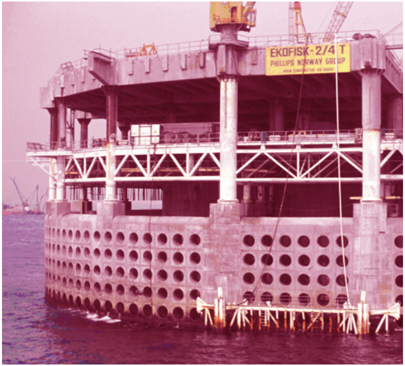


Photo: Norwegian Petroleum Museum/ConocoPhillips

Motivation

Subsidence of Venice

cased by the extraction of the aquifer.

Deformable porous media

The term deformable porous media refers to a deformable framework made of somewhat rigid material containing open spaces. The changes in pressure caused by motion, removal or addition of liquid or gas may cause deformations in the structure holding the fluid.

Modeling porous media systems
aims to describe the changes in a permeable material due to changes of the fluid it held.

In the past, modelers tended to ignore the geomechanics in their calculations.

Side effects of drilling:

consolidation - reduction in volume due to fluid extraction
compaction - reduction in volume due to air removal
subsidence - collapse

Poroelasticity models

Poroelasticity refers to fluid flow within a deformable porous medium under the assumption of relatively small deformations.

Examples of poroelastic structures

soil, rock, cartilage, brain, heart, bone

Various environmental, energy industry and biomechanics applications

- Subsidence, reservoir compaction
- Well stability, sand production
- Waste disposal
- Sequestration of carbon in saline aquifers
- Estimate tumor-induced stress levels in the brain
- Development of prosthetic devices for cartilage, bone, heart valves

Biot's consolidation model

primary variables: displacement $\mathbf{u} = (u_1, u_2, u_3)$ and fluid pressure p

Balance of linear momentum

$$-\nabla \cdot (\sigma(\mathbf{u}) - \alpha p I) = \mathbf{f}$$

$\sigma(\mathbf{u})$ - effective stress tensor, linear elastic, $\mathbf{f}$ - body force

Mass conservation

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot \mathbf{v_f} + s_f$$

η - fluid content , $\mathbf{v_f}$ - flux of fluid, s_f - volumetric fluid source term

Darcy's law for the flux of fluid

$$\mathbf{v_f} = -\frac{1}{\mu_f} K(\nabla p - \rho_f \mathbf{g})$$

K - permeability tensor

Stress-dependent permeability

In stress-sensitive reservoirs, variation of the effective stress resulting from fluid production may induce deformation of the rocks and cause permeability reduction. This effect may significantly reduced expected productivity.

We consider the permeability tensor, K, being dependent on the effective mean stress, σ_m.

Summary of equations

Coupling equations

- In the domain, at any time t

$$-\nabla \cdot (\sigma(\mathbf{u}) - \alpha p I) = \mathbf{f}$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha \nabla \cdot \mathbf{u}) - \frac{1}{\mu_f} \nabla \cdot K(\nabla p - \rho_f \mathbf{g}) = s_f$$

- Initial conditions at time $t = 0$

$$p(0) = p_0, \quad \mathbf{u}(0) = \mathbf{u}_0$$

- Plus adequate boundary conditions

For the mixed formulation for the flow, we introduce the variable

$$\mathbf{z} = -\frac{1}{\mu_f} K(\nabla p - \rho_f \mathbf{g}).$$

Variational problem

Find $\mathbf{u} \in \mathbf{u}_d + H^1([0,T];\mathbf{V}_0)$, $p \in H^1([0,T];L^2(\Omega))$ and $\mathbf{z} \in \mathbf{z}_d + L^2([0,T];\mathbf{S}_0)$ such that

$$a_{\mathbf{u}}(\mathbf{u},\mathbf{v}) - \alpha(\nabla\cdot\mathbf{v},p) = \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\Gamma_N} \mathbf{r}_N\cdot\mathbf{v},$$

$$\left(c_0\frac{\partial p}{\partial t}, w\right) + \alpha\left(\frac{\partial}{\partial t}\nabla\cdot\mathbf{u}, w\right) + (\nabla\cdot\mathbf{z}, w) = \int_\Omega s_f w,$$

$$(\mu_f K^{-1}\mathbf{z},\mathbf{s}) - (p,\nabla\cdot\mathbf{s}) = -\int_{\Gamma_p} p_D\mathbf{s}\cdot\eta + \int_\Omega \rho_f\mathbf{g}\cdot\mathbf{s}$$

holds for all $(\mathbf{v}, w, \mathbf{s}) \in (\mathbf{V}_0, L^2(\Omega), \mathbf{S}_0)$ and $t \in [0,T]$.

Semi-discrete approximation

$\mathcal{E}_h$ and $\mathcal{E}_H$ be two nondegenerate partitions of the polyhedral domain Ω with maximal element diameter h and H, respectively.

Mixed spaces for flow variables

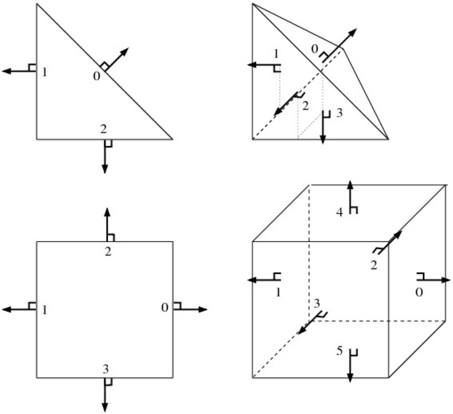
Examples of mixed spaces with the needed properties are the Raviart-Thomas-Nedelec spaces.

Example:
Lowest order Raviart-Thomas

2D

triangles: in each element
$\bar{p}$ =const, $\bar{\mathbf{z}} = (a + bx, c + by)^t$

quadrilaterals: in each element
$\bar{p}$ =const, $\bar{\mathbf{z}} = (a + bx, c + dy)^t$



Fully discrete approximation

$\Delta t = T/N$, where N denotes the number of time steps
$t_j = j\Delta t$

Find $\bar{\mathbf{u}}_j \in \bar{\mathbf{u}}_{d,j} + \mathbf{V}_{h,0}$, $\bar{p}_j \in W_h$, $\bar{\mathbf{z}}_j \in \bar{\mathbf{z}}_{d,j} + \mathbf{V}_{h,0}$ such that

$$\begin{aligned} a_{\mathbf{u}}(\bar{\mathbf{u}}_{j,\theta}, \mathbf{v}) - \alpha(\bar{p}_{j,\theta}, \nabla \cdot \mathbf{v}) &= \ell_{1j,\theta}(\mathbf{v}), \\ \left(c_0 \frac{\bar{p}_{j+1} - \bar{p}_j}{\Delta t}, w\right) + \alpha\left(\nabla \cdot \frac{\bar{\mathbf{u}}_{j+1} - \bar{\mathbf{u}}_j}{\Delta t}, w\right) + (\nabla \cdot \bar{\mathbf{z}}_{j,\theta}, w) &= \ell_{2j,\theta}(w), \\ (\mu_f K^{-1} \bar{\mathbf{z}}_{j,\theta}, \mathbf{s}) - (\bar{p}_{j,\theta}, \nabla \cdot \mathbf{s}) &= \ell_{3j,\theta}(\mathbf{s}), \end{aligned}$$

for all $(\mathbf{v}, w, \mathbf{s}) \in (\mathbf{V}_{h,0}, W_h, \mathbf{S}_{h,0})$.

$g_{j,\theta} = \frac{1}{2}(1+\theta)g_{j+1} + \frac{1}{2}(1-\theta)g_j$

- Implicit Euler method if $\theta = 1$
- Crank-Nicolson method if $\theta = 0$

A priori error estimate

$\bar{\mathbf{u}}$ - piecewise linear elements
$\bar{p}$ and $\bar{\mathbf{z}}$ - RT0 elements

Implicit Euler method

Theorem

If Δt small enough, there exists $C > 0$ such that

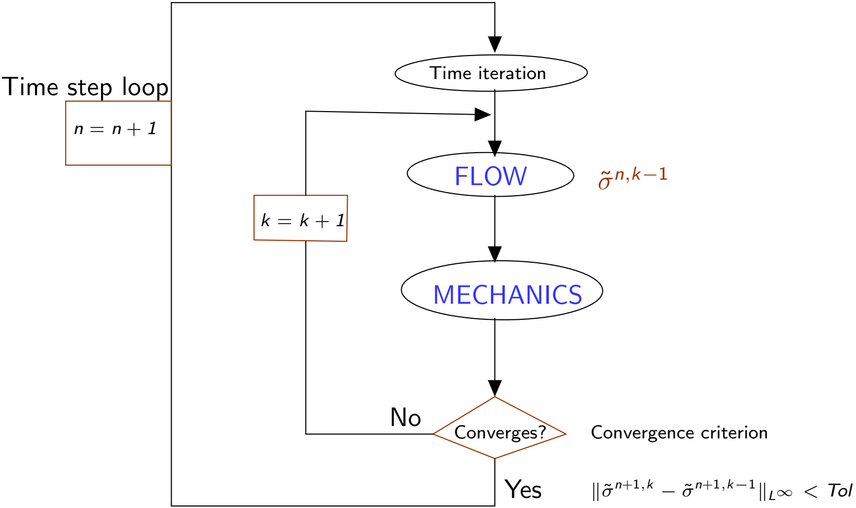
$$\|\mathbf{u} - \bar{\mathbf{u}}\|^2_{L^\infty(H^1)} + \|p - \bar{p}\|^2_{L^\infty(L^2)} + \|\mathbf{z} - \bar{\mathbf{z}}\|^2_{L^2(L^2)} \leq C(H^2 + h^2) + \mathcal{O}(\Delta t^2).$$

[Philips and Wheeler, Comput. Geosci., 2007]
[SB and Wheeler, Comput. Geosci., 2010] - stress-dependent permeability (nonlinear term)

If we use the Crank-Nicolson method, holds

$$\|\mathbf{u} - \bar{\mathbf{u}}\|^2_{L^\infty(H^1)} + \|p - \bar{p}\|^2_{L^\infty(L^2)} + \|\mathbf{z} - \bar{\mathbf{z}}\|^2_{L^2(L^2)} \leq C(H^2 + h^2) + \mathcal{O}(\Delta t^4).$$

Iterative coupling



Iterative coupling is stable and accurate. [Wheeler and Gai, Numer. Meth. PDEs, 2007]

Fully, iteratively, explicit and loosely coupled

Fully coupled The coupled governing equations of flow and geomechanics are solved simultaneously at every time step.

Iteratively coupled Either the flow, or mechanical, problem is solved first, then the other problem is solved using the intermediate solution information. This sequential procedure is iterated at each time step until the solution converges to within an acceptable tolerance. The converged solution is identical to that obtained using the fully coupled approach.

Explicitly coupled This is a special case of the iteratively coupled method, where only one iteration is taken.

Loosely coupled The coupling between the two problems is resolved only after a certain number of flow time steps. This method can save computational cost compared to the other strategies, but it is less accurate and requires reliable estimates of when to update the mechanical response.

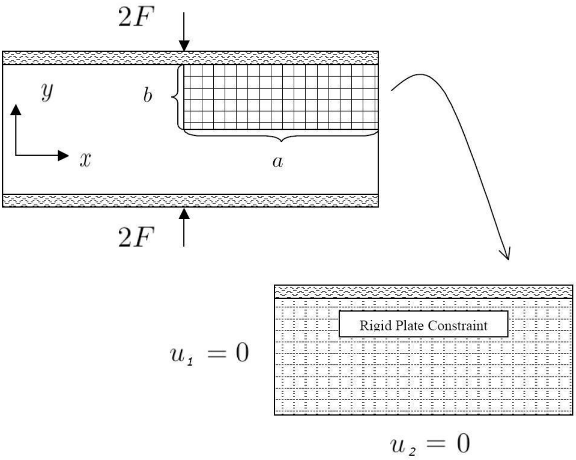
[Kim, Tchelepi, Juanes, SPE, 2009]

Mandel's problem

Mandel's solution has been used as a benchmark problem for testing the validity of numerical codes of poroelasticity.

Mandel, 1953 - analytical solution for pressure

Abousleiman et al., 1996 - analytical solution for displacement and stress

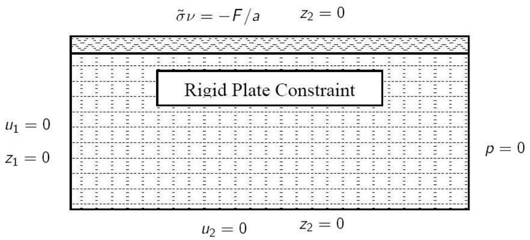


Mandel's problem

$$
\begin{aligned}
-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\nabla^2\mathbf{u}+\alpha\nabla p &= 0 \text{ in } \Omega\times(0,T] \\
\frac{\partial}{\partial t}(c_0 p+\alpha\nabla\cdot\mathbf{u})-\frac{1}{\mu_f}\nabla\cdot K\nabla p &= 0 \text{ in } \Omega\times(0,T]
\end{aligned}
$$

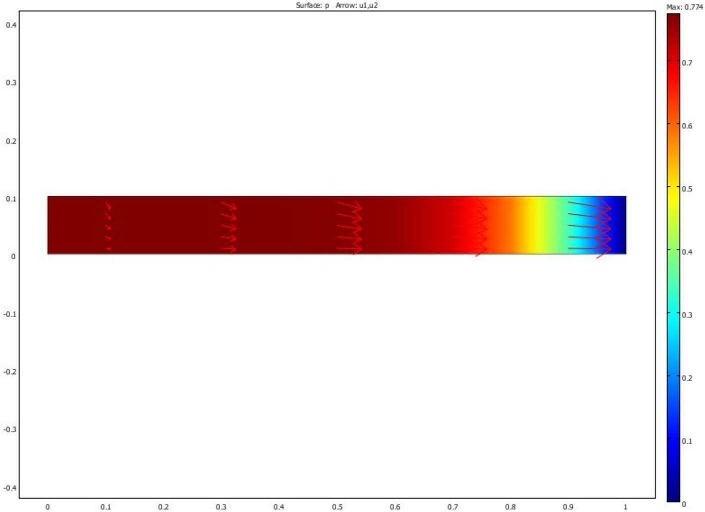
boundary conditions

$$
\begin{aligned}
& p=0,\ x=a, \quad -\frac{1}{\mu_f}K\nabla p\cdot\eta=0,\ x=0,\ y=0,\ y=b, \\
& u_1=0,\ x=0, \quad u_2=0,\ y=0, \\
& \frac{\partial u_y}{\partial x}=0,\ y=b, \\
& \tilde{\sigma}\eta=(-F/a)\eta,\ y=b, \quad \tilde{\sigma}\eta=0,\ x=0,\ x=a,\ y=0,
\end{aligned}
$$



Mandel's problem

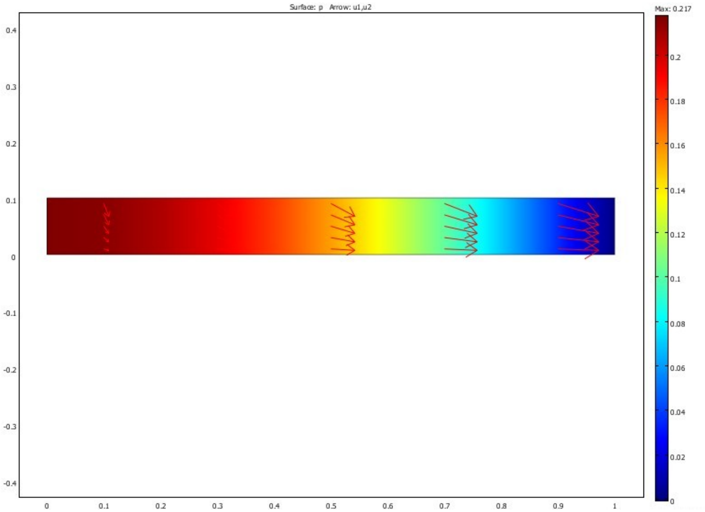
surface: p, arrows: u_1 and u_2



$T = 0.001$

Mandel's problem

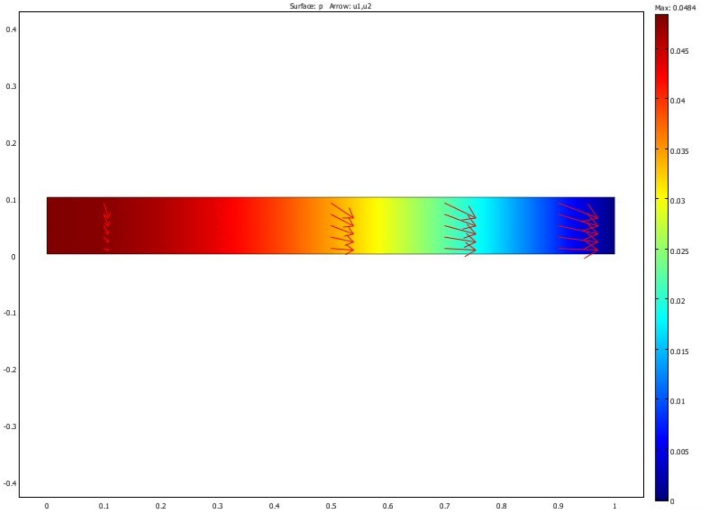
surface: p, arrows: u_1 and u_2



$T = 0.1$

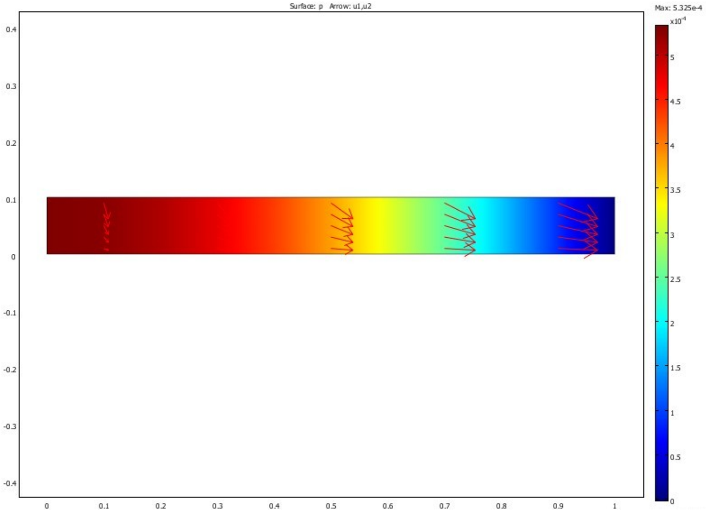
Mandel's problem

surface: p, arrows: u_1 and u_2



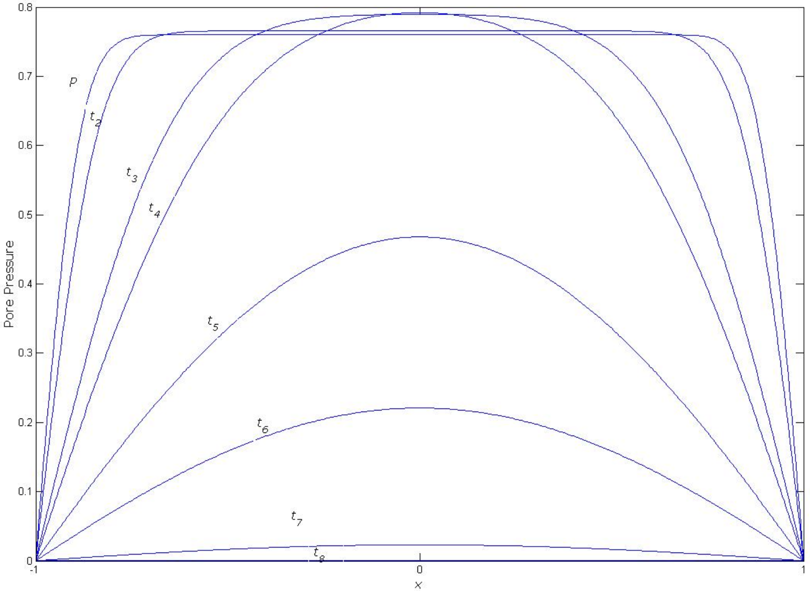
$T = 0.2$

surface: p, arrows: u_1 and u_2



$T = 0.5$

Mandel's problem



Mandel's problem

Numerical results

H	$\|\mathbf{u}-\bar{\mathbf{u}}\|_{H^1}$	rate	$\|p-\bar{p}\|_{L^2}$	rate	$\|\mathbf{z}-\bar{\mathbf{z}}\|_{L^2}$	rate
5.000e-2	1.222e-3	1.39	1.389e-2	1.53	2.416e-1	1.35
2.500e-2	4.653e-4	1.19	4.798e-3	1.21	9.452e-2	0.94
1.667e-2	2.878e-4	1.05	2.933e-3	1.03	6.453e-2	1.14
1.250e-2	2.130e-4	1.10	2.179e-3	1.08	4.654e-2	0.82
1.000e-2	1.665e-4	0.89	1.711e-3	0.92	3.875e-2	1.14
8.333e-3	1.415e-4	-	1.446e-3	-	3.149e-2	-

Convergence rates: bilinear elements for **u**, lowest order Raviart-Thomas space for p and **z**

Numerical results

H	$\|\mathbf{u}-\bar{\mathbf{u}}\|_{H^1}$	rate	$\|\mathbf{u}-\bar{\mathbf{u}}\|_{L^2}$	rate
5.000e-002	1.222e-3	1.39	2.773e-005	2.85
2.500e-002	4.653e-4	1.19	3.858e-006	2.33
1.667e-002	2.878e-4	1.05	1.499e-006	1.78
1.250e-002	2.130e-4	1.10	8.992e-007	2.66
1.000e-002	1.665e-4	-	4.965e-007	-

$\Longrightarrow$ Higher order of $\bar{\mathbf{u}}$ in the L^2-norm.

Summary

- The reliability of predictions depends on how well the models describe the physical problems.
- The numerical discretization of this model gives rise to algorithmic challenges including the issues of stability and accuracy of spatial and temporal discretizations.
- Our focus is to discuss the development of efficient and accurate numerical solutions, and give insight into the theoretical basis of the underlying methods.
- The applications of this type of models is far beyond Geophysics. There are lot of applications that use the same type of equations in a diverse range of fields.

"I really enjoy developing efficient and accurate solutions to real-world problems, while maintaining a solid theoretical base."

Mary Wheeler